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Technical Notes

A note on the Mass Transfer in the entrance to a flat duct with permeable walls

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NOMENCLATURE

D	diffusion coefficient
f_1, f_2, F, G	functions of ξ
h	channel height
k_w	mass transfer coefficient at the wall
N_{Sh_w}	wall Sherwood number, $k_w h/D$
$\bar{v}$	mean axial velocity
x	axial coordinate nondimensionalized with respect to $\bar{v}h^2/D$
y	transverse coordinate nondimensionalized with respect to h.

Greek symbols

$\Gamma(a)$	gamma function
$\Gamma(a, b)$	incomplete gamma function
ε	quantity defined by equation (10)
θ	dimensionless concentration
ξ	quantity defined by equation (10)
ω	dimensionless coordinate, $\frac{1}{2} - y$.

THE PROBLEM considered here is that in which a dilute solution with constant physical properties flows in fully developed laminar motion in a semi-infinite flat duct. The concentration θ is uniform up to a point ($x = 0$) where the fluid comes into contact with a permeable wall, outside of which the concentration is constant. The aim of this note is to find the concentration distribution and mass flux near $x = 0$. There are many applications of this work, for example in connection with the determination of the concentration of oxygen captured by blood flowing in tubes with permeable walls. This problem is similar to the classical 'Graetz problem' which has been studied by a number of investigators and these have been reviewed by Colton *et al.* [1], Walker and Davies [2] and Papoutsakis *et al.* [3]. The purpose of this note is to offer a second-order correction to the Leveque approach employed by Colton *et al.*

Consider a fluid with constant properties flowing between parallel plates which are a distance h apart and x and y are the coordinates parallel and perpendicular to the channel walls, respectively. It is assumed that the flow is steady, laminar and fully developed and the longitudinal mass diffusion is neglected in comparison with the lateral mass diffusion. The problem can therefore be stated in dimensionless form as, see Colton *et al.*

$$\frac{3}{2}(1-4y^2)\frac{\partial \theta}{\partial x} = \frac{\partial^2 \theta}{\partial y^2} \tag{1}$$

subject to the boundary conditions

$$\theta = 1, \quad x < 0, \quad \text{all } y \tag{2}$$

$$\frac{\partial \theta}{\partial y} = 0, \quad \text{all } x, \quad y = 0 \tag{3}$$

$$\frac{\partial \theta}{\partial y} = -N_{Sh_w}\theta, \quad x < 0, \quad y = \pm\tfrac{1}{2} \tag{4}$$

where N_{Sh_w} ($= k_w h/D$ and k_w is the mass transfer coefficient at the wall, h is the channel height and D the diffusion coefficient) is the Sherwood number. $N_{Sh_w} = 0$ is the constant flux boundary condition and $N_{Sh_w} = \infty$ the constant concentration boundary condition.

In general the separation of variables method can be employed to find the solution of equation (1) subject to conditions (2)–(4) but when x is very small a large number of terms is required in the series solution.

Hence near $x = 0$ and $y = \frac{1}{2}$ new independent variables x and $\omega (= \frac{1}{2} - y)$ are used. Thus equations (1)–(4) become

$$6\omega\underline{(1-\omega)}\frac{\partial \theta}{\partial x} = \frac{\partial^2 \theta}{\partial \omega^2} \tag{5}$$

$$\theta = 1, \quad x < 0, \quad \text{all } \omega \tag{6}$$

$$\theta = 1, \quad \text{all } x, \quad \omega \to \infty \tag{7}$$

$$\frac{\partial \theta}{\partial \omega} = N_{Sh_w}\theta, \quad \text{all } x, \quad \omega = 0. \tag{8}$$

These equations are the same as those given by Colton *et al.* except for the term underlined. In the Leveque analysis employed by Colton *et al.* [1] the curvature of the velocity profile near the wall is ignored so only the first term in their solution is correct and that term was also given by Bird [4].

A solution of equation (5) is looked for in the form

$$\theta = 1 + \sum_{n=1}^{\infty} \varepsilon^n f_n(\xi) \tag{9}$$

where

$$\varepsilon = 6^{-1/3} N_{Sh_w} x^{1/3} \quad \text{and} \quad \xi = 6^{1/3}\omega x^{-1/3}. \tag{10}$$

Substitution of expression (9) into equation (5) and collecting up the same powers of ε gives

$$f_1'' + \tfrac{1}{3}\xi^2 f_1' - \tfrac{1}{3}\xi f_1 = 0 \tag{11}$$

$$f_2'' + \tfrac{1}{3}\xi^2 f_1' - \tfrac{2}{3}\xi f_2 = \underline{-\frac{1}{3N_{Sh_w}}\xi^2(f_1 - \xi f_1')} \tag{12}$$

etc.

and the boundary conditions (6)–(8) become

$$f_1'(0) = 1, \quad f_1(\infty) = 0 \tag{13}$$

$$f_2'(0) = f_1(0), \quad f_2(\infty) = 0 \quad \text{etc.} \tag{14}$$

The term underlined in equation (12) is the correction to the work of Colton *et al.*, which allows for the curvature of the velocity profile near the wall. Equation (11) was given by Bird and Colton *et al.* and the solution is

$$f_1 = -9^{1/3}\left[\frac{\exp(-\xi^3/9)}{\Gamma(2/3)} - \frac{\xi}{9^{1/3}}\left\{1 - \frac{\Gamma(2/3,\xi^3/9)}{\Gamma(2/3)}\right\}\right]. \quad (15)$$

Thus the f_1 term alone corresponds to a Leveque-type solution for a flat duct with a constant flux at the wall. Hence the f_2 term is the first-order correction to the constant flux solution for finite nonzero N_{Sh_w}.

Writing $f_2(\xi) = F(\xi)+(1/N_{Sh_w})G(\xi)$ then equation (12) becomes

$$F''+\tfrac{1}{3}\xi^2F'-\tfrac{2}{3}\xi F = 0 \quad (16)$$

$$G''+\tfrac{1}{3}\xi^2G'-\tfrac{2}{3}\xi G = -\tfrac{1}{3}\xi^2(f_1-\xi f_1') \quad (17)$$

and boundary conditions (14) become

$$F'(0) = f_1(0), \quad F(\infty) = 0 \quad (18)$$

$$G'(0) = 0, \quad G(\infty) = 0. \quad (19)$$

Equations (16) and (17) subject to the boundary conditions (18) and (19) were solved numerically on the DEC 10 computer at Leeds using an Adams method. It was found that

$$F(0) = 2.0843, \quad G(0) = -0.41686. \quad (20)$$

The solutions for f_1, F and G are shown in Fig. 1.

The Sherwood numbers of interest can now be calculated. In particular the fluid side relation becomes

$$N_{Sh_w} = \frac{1}{1-\theta_w}\left(\frac{\partial\theta}{\partial\omega}\right)_{\omega=0} = 1.1829x^{-1/3}$$
$$-0.1167N_{Sh_w} - \underline{0.1767} + O(x^{1/3}). \quad (21)$$

The term underlined is the correction to the expression given by Colton *et al.* and further Colton *et al.* give the coefficient of the term in N_{Sh_w} to be -0.1320.

At small values of N_{Sh_w} the correction term presented here will dominate over the N_{Sh_w} term. The first term in expression (21) is the constant flux at the wall solution, the second term is the first-order correction for a linear velocity distribution whereas the third term is the first-order correction for the curvature of the velocity profile.

In conclusion the general comments made by Colton *et al.* are still valid but expression (21) should be used rather than their expression (63) for N_{Sh_r} at small values of x. At large values of x their series solution is the appropriate one to use.

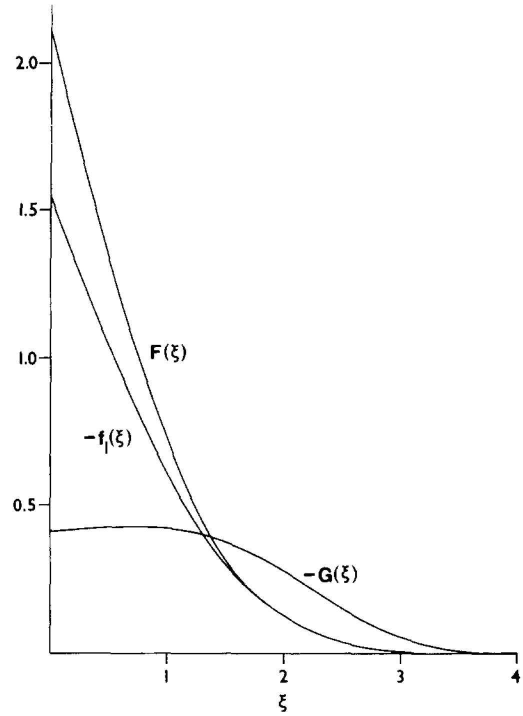

FIG. 1. f_1, F and G as functions of ξ.

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